Addendum: Vectorial diffraction

11.6. Detailled development of Bouwkamp.

We start from the differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + k^2 E_z = 0 \tag{1}$$

where we introduce the following changing of variable:

$$\begin{cases} u = 2r\left[\cos\left(\frac{\varphi}{2}\right)\right]^2 \\ v = -r\cos\varphi \end{cases} \quad \text{or} \quad \begin{cases} u = r(1+\cos\varphi) \\ v = -r\cos\varphi \end{cases} \tag{2}$$

Let us develop the general differential operators:

$$\frac{\partial}{\partial r} = \frac{\partial u}{\partial r}\frac{\partial}{\partial u} + \frac{\partial v}{\partial r}\frac{\partial}{\partial v} \tag{3}$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial u}{\partial \varphi}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \varphi}\frac{\partial}{\partial v} \tag{4}$$

Obviously:

$$\frac{\partial u}{\partial r} = 1 + \cos\varphi \qquad \frac{\partial v}{\partial r} = -\cos\varphi$$
$$\frac{\partial u}{\partial \varphi} = -r\sin\varphi \qquad \frac{\partial v}{\partial \varphi} = r\sin\varphi \tag{5}$$

Now we expressed these derivatives in terms of u and v:

First:

$$u + v = r + r\cos\varphi - r\cos\varphi = r \tag{6}$$

$$\cos\varphi = -\frac{v}{r} = -\frac{v}{u+v} \tag{7}$$

$$\sin\varphi = \sqrt{1-(\cos\varphi)^2} = \sqrt{1-\frac{v^2}{(u+v)^2}} = \frac{\sqrt{u(u+2v)}}{u+v} \tag{8}$$

then we get:

$$\begin{cases} M = \frac{\partial u}{\partial r} = 1 + \cos\varphi = 1 - \frac{v}{u+v} = \frac{u}{u+v} \\ N = \frac{\partial v}{\partial r} = -\cos\varphi = \frac{v}{u+v} \\ P = \frac{\partial u}{\partial \varphi} = -r\sin\varphi = -(u+v)\frac{\sqrt{u(u+2v)}}{u+v} = -\sqrt{u(u+2v)} \\ Q = \frac{\partial v}{\partial \varphi} = r\sin\varphi = \sqrt{u(u+2v)} \end{cases} \tag{9}$$

into:

$$\frac{\partial}{\partial r} = M\frac{\partial}{\partial u} + N\frac{\partial}{\partial v} \tag{10}$$

and

$$\frac{\partial}{\partial \varphi} = P\frac{\partial}{\partial u} + Q\frac{\partial}{\partial v} \tag{11}$$

Next we work on $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)$.

$$\frac{\partial\phi}{\partial r} = M\frac{\partial\phi}{\partial u} + N\frac{\partial\phi}{\partial v} \tag{12}$$

$$r\frac{\partial\phi}{\partial r} = rM\frac{\partial\phi}{\partial u} + rN\frac{\partial\phi}{\partial v} = (u+v)M\frac{\partial\phi}{\partial u} + (u+v)N\frac{\partial\phi}{\partial v} \tag{13}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = M\frac{\partial}{\partial u}\left(r\frac{\partial\phi}{\partial r}\right) + N\frac{\partial}{\partial v}\left(r\frac{\partial\phi}{\partial r}\right) \tag{14}$$

$$= M\frac{\partial}{\partial u}\left[(u+v)M\frac{\partial\phi}{\partial u} + (u+v)N\frac{\partial\phi}{\partial v}\right] + N\frac{\partial}{\partial v}\left[(u+v)M\frac{\partial\phi}{\partial u} + (u+v)N\frac{\partial\phi}{\partial v}\right] \tag{15}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = {} & M\frac{\partial}{\partial u}[(u+v)M]\frac{\partial\phi}{\partial u} + (u+v)M^2\frac{\partial^2\phi}{\partial u^2} \\ & + M\frac{\partial}{\partial u}[(u+v)N]\frac{\partial\phi}{\partial v} + (u+v)MN\frac{\partial^2\phi}{\partial u\partial v} \\ & + N\frac{\partial}{\partial v}[(u+v)M]\frac{\partial\phi}{\partial u} + (u+v)MN\frac{\partial^2\phi}{\partial v\partial u} \\ & + N\frac{\partial}{\partial v}[(u+v)N]\frac{\partial\phi}{\partial v} + (u+v)N^2\frac{\partial^2\phi}{\partial v^2}\end{aligned} \tag{16}$$

Because

$$\begin{aligned}\frac{\partial}{\partial u}[(u+v)M] &= \frac{\partial}{\partial u}\left[(u+v)\frac{u}{u+v}\right] = \frac{\partial u}{\partial u} = 1 \\ \frac{\partial}{\partial u}[(u+v)N] &= \frac{\partial}{\partial u}\left[(u+v)\frac{v}{u+v}\right] = \frac{\partial v}{\partial u} = 0 \\ \frac{\partial}{\partial v}[(u+v)M] &= \frac{\partial}{\partial v}\left[(u+v)\frac{u}{u+v}\right] = \frac{\partial u}{\partial v} = 0 \\ \frac{\partial}{\partial v}[(u+v)N] &= \frac{\partial}{\partial v}\left[(u+v)\frac{v}{u+v}\right] = \frac{\partial v}{\partial v} = 1\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{1}{u+v}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = {} & \frac{M}{u+v}\frac{\partial\phi}{\partial u} + M^2\frac{\partial^2\phi}{\partial u^2} + 0 + MN\frac{\partial^2\phi}{\partial u\partial v} \\ & + 0 + MN\frac{\partial^2\phi}{\partial v\partial u} + \frac{N}{u+v}\frac{\partial\phi}{\partial v} + N^2\frac{\partial^2\phi}{\partial v^2}\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = {} & \frac{u}{(u+v)^2}\frac{\partial\phi}{\partial u} + \frac{u^2}{(u+v)^2}\frac{\partial^2\phi}{\partial u^2} + \frac{2uv}{(u+v)^2}\frac{\partial^2\phi}{\partial u\partial v} \\ & + \frac{v}{(u+v)^2}\frac{\partial\phi}{\partial v} + \frac{v^2}{(u+v)^2}\frac{\partial^2\phi}{\partial v^2}\end{aligned} \tag{19}$$

Next we work on $\frac{\partial^2\phi}{\partial\varphi^2}$.

$$\frac{\partial\phi}{\partial\varphi} = P\frac{\partial\phi}{\partial u} + Q\frac{\partial\phi}{\partial v} \tag{20}$$

$$\frac{\partial^2\phi}{\partial\varphi^2} = \frac{\partial}{\partial\varphi}\left[P\frac{\partial\phi}{\partial u} + Q\frac{\partial\phi}{\partial v}\right]$$

$$= P\frac{\partial}{\partial u}\left(P\frac{\partial\phi}{\partial u}\right) + Q\frac{\partial}{\partial v}\left(P\frac{\partial\phi}{\partial u}\right) \tag{21}$$

$$+ P\frac{\partial}{\partial u}\left(Q\frac{\partial\phi}{\partial v}\right) + Q\frac{\partial}{\partial v}\left(Q\frac{\partial\phi}{\partial v}\right)$$

$$\frac{\partial^2\phi}{\partial\varphi^2} = P\frac{\partial P}{\partial u}\frac{\partial\phi}{\partial u} + P^2\frac{\partial^2\phi}{\partial u^2} + Q\frac{\partial P}{\partial v}\frac{\partial\phi}{\partial u} + PQ\frac{\partial^2\phi}{\partial u\partial v}$$

$$+ P\frac{\partial Q}{\partial u}\frac{\partial\phi}{\partial u} + PQ\frac{\partial^2\phi}{\partial v\partial u} + Q\frac{\partial Q}{\partial v}\frac{\partial\phi}{\partial v} + Q^2\frac{\partial^2\phi}{\partial v^2} \tag{22}$$

Because

$$\frac{\partial P}{\partial u} = \frac{\partial}{\partial u}(-\sqrt{u(u+2v)}) = -\frac{2u+2v}{2\sqrt{u(u+2v)}} = -\frac{u+v}{\sqrt{u(u+2v)}} \tag{23}$$

$$\frac{\partial P}{\partial v} = \frac{\partial}{\partial v}(-\sqrt{u(u+2v)}) = -\frac{2u}{2\sqrt{u(u+2v)}} = -\frac{u}{\sqrt{u(u+2v)}} \tag{24}$$

$$\frac{\partial Q}{\partial u} = \frac{u+v}{\sqrt{u(u+2v)}} \qquad \text{and} \qquad \frac{\partial Q}{\partial v} = \frac{u}{\sqrt{u(u+2v)}} \tag{25}$$

after replacing:

$$\frac{\partial^2\phi}{\partial\varphi^2} = \frac{\sqrt{u(u+2v)}(u+v)}{\sqrt{u(u+2v)}}\frac{\partial\phi}{\partial u} + u(u+2v)\frac{\partial^2\phi}{\partial u^2}$$

$$-\frac{\sqrt{u(u+2v)}u}{\sqrt{u(u+2v)}}\frac{\partial\phi}{\partial u} - u(u+2v)\frac{\partial^2\phi}{\partial u\partial v}$$

$$-\frac{\sqrt{u(u+2v)}(u+v)}{\sqrt{u(u+2v)}}\frac{\partial\phi}{\partial v} - u(u+2v)\frac{\partial^2\phi}{\partial u\partial v} \tag{26}$$

$$+\frac{\sqrt{u(u+2v)}u}{\sqrt{u(u+2v)}}\frac{\partial\phi}{\partial v} + u(u+2v)\frac{\partial^2\phi}{\partial v^2}$$

$$\frac{\partial^2\phi}{\partial\varphi^2} = (u+v)\frac{\partial\phi}{\partial u} + u(u+2v)\frac{\partial^2\phi}{\partial u^2} - u\frac{\partial\phi}{\partial u} - u(u+2v)\frac{\partial^2\phi}{\partial u\partial v}$$

$$-(u+v)\frac{\partial\phi}{\partial v} - u(u+2v)\frac{\partial^2\phi}{\partial u\partial v} + u\frac{\partial\phi}{\partial v} + u(u+2v)\frac{\partial^2\phi}{\partial v^2} \tag{27}$$

finally:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}+k^2\phi \;=\; \frac{u}{(u+v)^2}\frac{\partial\phi}{\partial u}+\frac{u^2}{(u+v)^2}\frac{\partial^2\phi}{\partial u^2}+\frac{2uv}{(u+v)^2}\frac{\partial^2\phi}{\partial u\partial v}+\frac{v}{(u+v)^2}\frac{\partial\phi}{\partial v}$$

$$+\frac{v^2}{(u+v)^2}\frac{\partial^2\phi}{\partial v^2}+\frac{(u+v)}{(u+v)^2}\frac{\partial\phi}{\partial u}+\frac{u(u+2v)}{(u+v)^2}\frac{\partial^2\phi}{\partial u^2}-\frac{u}{(u+v)^2}\frac{\partial\phi}{\partial u}-\frac{u(u+2v)}{(u+v)^2}\frac{\partial^2\phi}{\partial u\partial v} \tag{28}$$

$$-\frac{(u+v)}{(u+v)^2}\frac{\partial\phi}{\partial v}-\frac{u(u+2v)}{(u+v)^2}\frac{\partial^2\phi}{\partial u\partial v}+\frac{u}{(u+v)^2}\frac{\partial\phi}{\partial v}+\frac{u(u+2v)}{(u+v)^2}\frac{\partial^2\phi}{\partial v^2}+k^2\phi$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2}+k^2\phi \;=$$

$$\frac{2u}{(u+v)}\frac{\partial^2\phi}{\partial u^2}-\frac{2u}{(u+v)}\frac{\partial^2\phi}{\partial u\partial v}+\frac{\partial^2\phi}{\partial v^2}+\frac{1}{(u+v)}\frac{\partial\phi}{\partial u}+k^2\phi \tag{29}$$