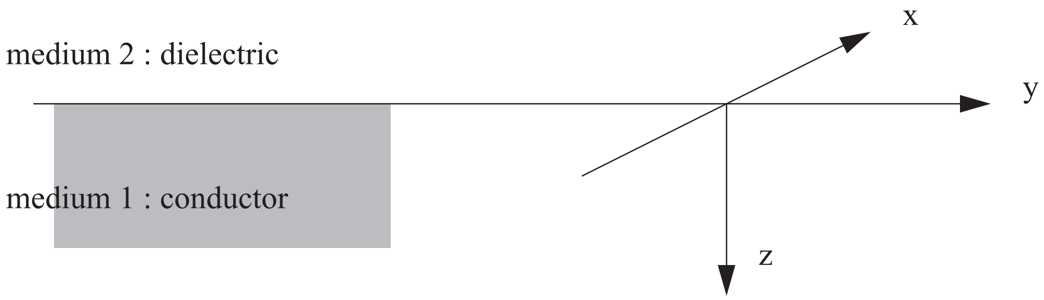
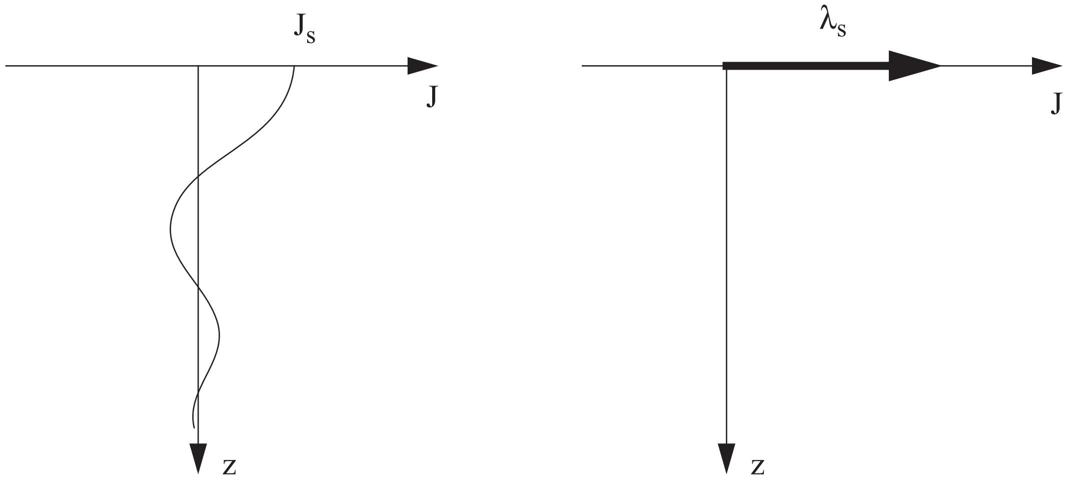
Addendum: Linking fields and Skin-effect currents.

All waveguides build with metallic walls show some attenuation. A part of it is due to the finite conductivity of the metal of the walls. Here again, the current flowing in the conductor follow the theory of the Skin-effect.

The theory of the Skin-effect describe clearly how the current density **J** (expressed in A/m^2) is distributed in the conductor as function of the value of the material parameters μ and σ and the frequency f.
When the conductivity σ goes to infinity, the conductor is said to be perfect or ideal and the skin depth δ shrinks to zero (see formula of $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$).
In that case the current distribution becomes a Dirac function situated exactly at the interface between the dielectric and the conductor.
The configuration of the axis are (look out: it is different from Chapter 4,/1):



The next figure shows the two cases: real conductor and ideal conductor:



Real conductor, finite conductivity   Ideal conductor, infinite conductivity

Because the induced currents in the conductor flow in a real medium, a power dissipation occurs in this medium which produces the attenuation of the electromagnetic waves travelling through the waveguide.
The dissipation phenomenon that take place in the real conductor has been worked out in chapter 4.1. for a simplified situation where the current is the same all over the conductor (independant of the value of x and y.
This simplified situation does not weakens the validty of the conclusion about the dissipation calculus, indeed the skin depth is in most cases quite small so that the x and y dimensions of the waveguide are allways much larger than the z dimension of the Skin-effect phenomenon.
As such we may apply the expressions of the Skin-effect (current distribution and power dissipation) of Chapter 4.1 to various waveguide configurations.

However a special theoretical development has been made for the cylindrical conductor.

Making the link.

The next question is to link the amplitude of the induced currents flowing in the conductor to the electric and magnetic fields present above the conductor (in the waveguide).

In the ideal case, the equations expressing the boundary conditions permit to obtain the value of the dirac surface current λ_s:

$$\lambda_s = \mathbf{1}_n \wedge (\boldsymbol{H}_2 - \boldsymbol{H}_1)$$

where $\mathbf{1}_n$ is the unit vector $-\mathbf{1}_z$ and the indexes 1 and 2 correspond repectively to the conductor and the dielectric.

Again when the conductor is ideal, the electric and magnetic fields are allways zero: $\boldsymbol{H}_1 = o$.

So: $\lambda_s = \mathbf{1}_n \wedge \boldsymbol{H}_{ext}$

For instance if $\boldsymbol{H}_{ext} = H_{ext} \cdot \mathbf{1}_y$ it gives: $\lambda_s = -\mathbf{1}_z \wedge H_{ext} \cdot \mathbf{1}_y = H_{ext} \cdot \mathbf{1}_x$

Remark that the expression λ_s has the same dimension as the magnetic field H (in A/m) which shows clearly that it is an infinitively thin sheet of current.

In this case, it is of no use to estimate the dissipated power in the conductor because there is no, the conductor is assumed to be ideal.

In the real case, the equations expressing the boundary conditions may not be implemented to find the value of the dirac surface current λ_s, because this term is zero by definition.

But the same expression can be used to obtain the magnetic field in the conductor, just below the surface. $\boldsymbol{H}_{int} = \boldsymbol{H}_{ext}$

> **Question:** why dont we use the continuity expression of the tangential electrical field instead of the magnetic field to link both mediums?
> The answer is simply practical. In most cases the conductors are of good quality and the ideal approximation is extremely accurate for the determination of the fields in the dielectric medium that fills the waveguide. That means that these fields dont change significantly when the conductors behave real.
> The continuity expression of the tangential electrical field written for an ideal conductor always obliges the electrical filed to be zero on the conductor. This is not a practical start value for a theoretical step towards the estimation of expressions of the fields and currents in real conductors.

Now that we have the magnetic field in the conductor, we can find the electrtical field by applying the rot **H** expression: rot $\boldsymbol{H}_{int} = (\sigma + j\omega\varepsilon) \cdot \boldsymbol{E}_{int}$ to the magnetic field.

Therefore we assume that the fields penetrating into the conductor behaves like the fields described in Chapter 4.1 i.e.e like the case of the attenuated plane wave.

Indeed the mathematical form describing the current density distribution found in Chapter 4.1

$$\boldsymbol{J}(z) = J_s e^{-\sqrt{j\omega\mu\sigma} z} \cdot \mathbf{1}_x = J_s e^{-\gamma z} \cdot \mathbf{1}_x$$

can be extended to the electrical and magnetic fields.

This is obvious for E because $\boldsymbol{E} = \boldsymbol{J}/\sigma$ giving: $\boldsymbol{E} = E_0 e^{-j\gamma z} \cdot \mathbf{1}_x$

H is obtained by the expression rot $\boldsymbol{E} = -j\omega\mu \cdot \boldsymbol{H}$
which gives:

$$\boldsymbol{H} = -\frac{1}{j\omega\mu}\text{rot } \boldsymbol{E} = \frac{\gamma}{j\omega\mu}E_0 e^{-j\gamma z} \cdot \mathbf{1}_y = \frac{E_0}{\eta_c}e^{-j\gamma z} \cdot \mathbf{1}_y = H_0 \cdot e^{-j\gamma z}\mathbf{1}_y$$

Remark that for this development one has made no simplification in the Helmholz equation wich generalize also the coefficients

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \approx \sqrt{j\omega\mu\sigma}$$

and

$$\eta_c = \sqrt{\frac{j\omega\mu}{(\sigma + j\omega\varepsilon)}} \approx \sqrt{\frac{j\omega\mu}{\sigma}}$$

This is quite easy to work out for the reader; it is similar to the plane wave but with a strong attenuation. It is also completely worked out in the references:
C.A. Johnk Engineering Electromagnetic Fields and Waves Wiley 1988, Chapter 3.6.
C.A. Balanis Advanced Enginering Electromagnetics , Wiley 1989, Chapter 4.3.1.
See also the *Leontovitch impedance boundary condition* in Akira ISHIMARYU Electromagnetic Wave Propagation, Radiation, and Scattering , Prentice Hall 1991, p15.

So applying:

$$\boldsymbol{E}_{int} = \frac{1}{(\sigma + j\omega\varepsilon)}\text{rot } \boldsymbol{H}_{int} = \frac{1}{(\sigma + j\omega\varepsilon)}\text{rot } (H_0 e^{-j\gamma z} \cdot \mathbf{1}_y) = \eta_c \cdot H_{ext}e^{-j\gamma z} \cdot \mathbf{1}_x$$

Re-using expression $\boldsymbol{J} = \sigma\boldsymbol{E}$, we obtain the current density in the conductor

$$\boldsymbol{J}(z) = \sigma\eta_c \cdot H_{ext}e^{-j\gamma z} \cdot \mathbf{1}_x = \sigma\sqrt{\frac{j\omega\mu}{\sigma}} \cdot H_{ext}e^{-j\gamma z} \cdot \mathbf{1}_x = \sqrt{j\omega\mu\sigma} \cdot H_{ext}e^{-j\gamma z} \cdot \mathbf{1}_x$$

as a function of the external magnetic field.

This last equation gives the value of J_s:

$$J_s = \sqrt{j\omega\mu\sigma}H_{ext}$$

Another way of linking the current density flowing in the conductor to the external magnetic field is as follows: we concentrate all the current density into an infinitesimal thin sheet of surface current λ_s.
The sheet of current λ_s is then defined by the continuity condition:

$$\lambda_s = H_{ext} \cdot \mathbf{1}_x$$

Again we assume that the current density follows the general expression:

$$\boldsymbol{J}(z) = J_x e^{-\gamma z} \cdot \mathbf{1}_x$$

but where we replace coefficient J_s by an unknown coefficient J_x.

Like seen in Chapter 4.1 there is a relation between the surface current λ_s and J_x:

$$\lambda_s = \int_0^\infty \boldsymbol{J}(z)dy = \frac{1}{u}\int_0^u\int_0^\infty \boldsymbol{J}(z)dydu = \frac{1}{u}I\cdot\mathbf{1}_x = \frac{J_x\delta}{1+j}\cdot\mathbf{1}_x$$

which permits to get J_x:

$$J_x = \frac{1+j}{\delta}|\lambda_s| = \frac{1+j}{\delta}H_{ext} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}H_{ext} = \sqrt{j\omega\mu\sigma}H_{ext} = J_s$$

This shows that both methods are equivalent.

Power dissipated in the conductor.

The expression of the dissipated power calculated at Chapter 4.1 can be used:

$$P = \frac{R\cdot|I|^2}{2} = \frac{ul\delta}{4\sigma}\cdot J_s^{\,2}$$

Remark that this is the power dissipated in a rectangular patch of dimension $u \times l$, so we have to replace P by dP when we replace ul by $dxdy$. $\quad dP = \frac{\delta}{4\sigma}\cdot J_s^{\,2}\cdot dxdy$

Here again it is obvious that both expressions are equivalent:

$$dP = \frac{R\cdot|I|^2}{2} = \frac{R}{2}\cdot\text{Real}\ (I\cdot I^*) = \frac{1}{2}\cdot\frac{dx}{\sigma\delta dy}\cdot\text{Real}\ [\lambda_s dy\cdot(\lambda_s dy)^*]$$

which gives: $\quad dP = \frac{1}{2}\cdot\frac{1}{\sigma\delta}\cdot\text{Real}\ [\lambda_s\cdot(\lambda_s)^*]$

or again $\quad dP = \frac{\delta}{4\sigma}\cdot J_s^{\,2}\cdot dxdy$

Remark: one can calculate this dissipated power by another way; just by integrating the vector of Poynting of the attenuated plane wave that enters into the conductor.

$$dP = \frac{1}{2}\cdot\text{Real}\ [\boldsymbol{E}_{int}\wedge(\boldsymbol{H}_{int})^*]\cdot dxdy = \frac{1}{2}\cdot\text{Real}\ [\eta_c\boldsymbol{H}_{int}\cdot(\boldsymbol{H}_{int})^*]\cdot dxdy$$

or $\quad dP = \frac{1}{2}\cdot\text{Real}\ (\eta_c)\cdot|\boldsymbol{H}_{ext}|^2\cdot dxdy$

When worked out, $\quad \text{Real}\ (\eta_c) = \text{Real}\left(\sqrt{\frac{j\omega\mu}{\sigma}}\right) = \text{Real}\left(\frac{1+j}{\sqrt{2}}\right)\cdot\sqrt{\frac{\omega\mu}{\sigma}} = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\omega\mu}{\sigma}}$

this alternative solution delivers the same result.

$$dP = \frac{\delta}{4\sigma}\cdot J_s^{\,2}\cdot dxdy$$

This is also developed in Reference:

C.A. Balanis Advanced Enginering Electromagnetics , Wiley 1989, Chapter 4.3.1.