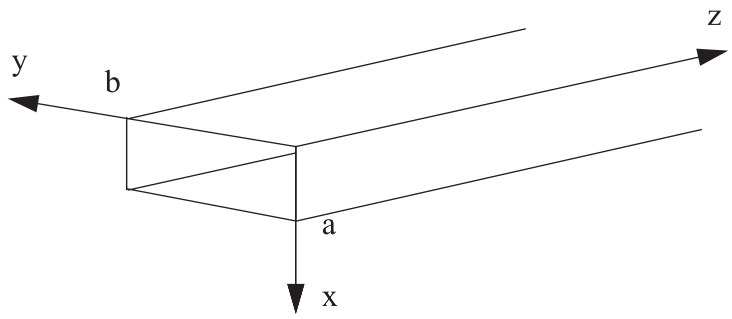
Addendum: Attenuation in a waveguides.

All waveguides show some attenuation. A part of it is due to the finite conductivity of the metal of the walls. The current flowing in the conductors (metal walls) follow the earlier explained theory of the Skin-effect.
Remark that another part of the attenuation comes from the loss in the dielectric that fills the waveguide.
A third and last part can be due to the use of the waveguide at a frequency below cutt-off frequency (see previously in chapter 8). This last case is unusual because it corresponds to a bad use of the waveguide. Nevertheless, some attenuators exploits this principle.

An example of a calculation of the attenuation due to the finite conductivity of the metal of the walls is developed below. We use the following set-up.



The calculation is based on the Skin-effect derivation that is presented in Addendum Skin Effect which boils down to the expression of the dissipated power .

$$dP = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{\omega\mu}{\sigma}} \cdot |\boldsymbol{H}_{ext}|^2 \cdot dxdy = \frac{1}{2} \cdot R_s \cdot |\boldsymbol{H}_{ext}|^2 \cdot dxdy$$

where R_s is the real part of the surface impedance

$$Z_s = R_s + jX_s = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}} \cdot (1 + j)$$

The developement starts from the fields of the TE01 mode in rectangular waveguide established previously in chapter 8:

$$E_x = A_{01} \cdot \sin\frac{\pi y}{b} \cdot e^{-\alpha z} \cdot e^{-j\Gamma z}$$

$$E_y = 0$$

$$E_z = 0$$

$$H_x = 0$$

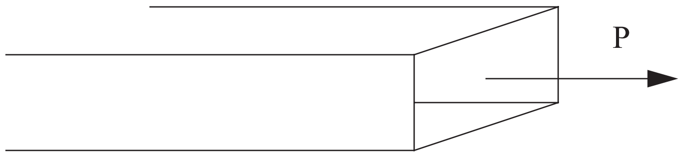
$$H_y = \frac{\Gamma}{\omega\mu} \cdot E_x$$

$$H_z = -j\frac{\pi}{b}\frac{A_{01}}{\omega\mu} \cdot \cos\frac{\pi y}{b} \cdot e^{-\alpha z} \cdot e^{-j\Gamma z}$$

with $c = \dfrac{c_0}{\sqrt{\varepsilon_r \mu_r}}$; $\Gamma = \sqrt{\left(\dfrac{\omega}{c}\right)^2 - \left(\dfrac{\pi}{b}\right)^2} = \dfrac{\omega}{c}\sqrt{1 - \left(\dfrac{f_{c,\,01}}{f}\right)^2}$; $f_c = f_{c,\,01} = \dfrac{c}{2b}$

and where the coefficient α of the exponent expresses the attenuation, but is still unknown.

Power traveling through the waveguide.
To derive the power transmitted by the waveguide, we simply integrate the vector of Poynting over the surface of the waveguide at any section z.



The vector of Poynting is by definition:

$$\boldsymbol{p} = \boldsymbol{E} \wedge \boldsymbol{H}^* = E_x \cdot \mathbf{1}_x \wedge (H_y \cdot \mathbf{1}_y + H_z \cdot \mathbf{1}_z)^* = E_x H_y^* \cdot \mathbf{1}_z - E_x H_z^* \cdot \mathbf{1}_y$$

which must be integrated:

$$P = \frac{1}{2} \cdot \int_{x=0}^{x=a} \int_{y=0}^{y=b} \boldsymbol{p} \cdot \mathbf{1}_z \cdot dxdy = \frac{1}{2} \cdot \int_{x=0}^{x=a} \int_{y=0}^{y=b} E_x H_y^* \cdot dxdy$$

or

$$P = \frac{1}{2} \cdot \int_{x=0}^{x=a} \int_{y=0}^{y=b} \frac{\Gamma}{\omega\mu}(A_{01})^2 \left(\sin\frac{\pi y}{b}\right)^2 \cdot e^{-2\alpha z} dxdy = \frac{{A_{01}}^2 \Gamma ab}{4\omega\mu} \cdot e^{-2\alpha z}$$

This can be rewritten as: $P = \frac{{A_{01}}^2 ab}{4\eta_{TE,\,01}}$ like in Ref: Johnk (see below)

If we define: $\eta_{TE,\,01} = \frac{\omega\mu}{\Gamma} = \frac{\eta}{\sqrt{1-\left(\frac{f_{c,\,01}}{f}\right)^2}}$ and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$, we can rewrite P as:

$$P = \frac{{A_{01}}^2 ab}{4\eta_{TE,\,01}} \cdot e^{-2\alpha z} = \frac{1}{4}\frac{{A_{01}}^2}{\eta} ab \sqrt{1-\left(\frac{f_{c,\,01}}{f}\right)^2} e^{-2\alpha z}$$

So the derivate of P to z gives the

$$\frac{\partial}{\partial z}P(z) = \frac{\partial}{\partial z}\left[\frac{{A_{01}}^2 ab}{4\eta_{TE,\,01}} \cdot e^{-2\alpha z}\right] = -2\alpha \cdot \frac{{A_{01}}^2 ab}{4\eta_{TE,\,01}} \cdot e^{-2\alpha z} = -2\alpha \cdot P(z)$$

where we can replace:

$$\frac{\partial}{\partial z}P(z) = \frac{dP(z)}{dz}$$

consequently

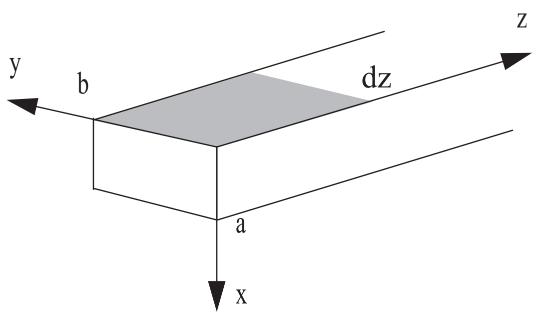
$$\alpha = -\frac{1}{2} \cdot \frac{\frac{dP(z)}{dz}}{P(z)}$$

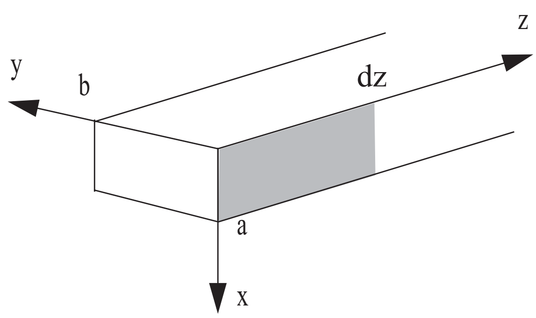
Dissipated power in the walls.

The dissipated power in the walls for a slice of dz of waveguide is derived as follows. We start with the power dissipated in an elementary surface which is then integrated over the four side walls. Remark that the minus sign expres that the transmitted power decraeses with increasing z.

$$dP(z) = -\int\limits_{\text{over the four walls}} \frac{1}{2} \cdot R_s \cdot |\boldsymbol{H}_{ext}|^2 \cdot \begin{Bmatrix} dx \\ \text{or} \\ dy \end{Bmatrix} dz$$

Because of symmetry, we may split this integral in twice the integral over only two walls:





$$\frac{dPz}{dz} = -2 \cdot \int_{x=0}^{x=a} \frac{1}{2} \cdot R_s \cdot \left| \left(-j\frac{\pi}{b}\frac{A_{01}}{\omega\mu} \cdot e^{-\alpha z} \cdot e^{-j\Gamma z} \right) \cdot \mathbf{1}_z \right|^2 dx$$

$$-\left(2 \cdot \int_{y=b}^{y=b} \frac{1}{2} \cdot R_s \cdot \left| \left(\Gamma\frac{A_{01}}{\omega\mu}\sin\frac{\pi y}{b} \cdot e^{-\alpha z}e^{-j\Gamma z} \right) \cdot \mathbf{1}_y + \left(-j\frac{\pi}{b}\frac{A_{01}}{\omega\mu}\cos\frac{\pi y}{b} \cdot e^{-\alpha z}e^{-j\Gamma z} \right) \cdot \mathbf{1}_z \right|^2 dy \right)$$

or:

$$\frac{dPz}{dz} = -\left(R_s \cdot \frac{\pi^2}{b^2}\left(\frac{A_{01}}{\omega\mu}\right)^2 \cdot e^{-2\alpha z} \cdot dz \cdot \int_{x=0}^{x=a} dx \right)$$

$$-\left(R_s \cdot \left(\frac{A_{01}}{\omega\mu}\right)^2 \cdot e^{-2\alpha z} \cdot dz \cdot \int_{y=b}^{y=b} \left[\Gamma^2\left(\sin\frac{\pi y}{b}\right)^2 + \frac{\pi^2}{b^2}\left(\cos\frac{\pi y}{b}\right)^2 \right] \cdot dy \right)$$

or

$$\frac{dPz}{dz} = -\left(R_s \cdot \left(\frac{A_{01}}{\omega\mu}\right)^2 \cdot e^{-2\alpha z} \cdot \left[\frac{\pi^2}{b^2}a + \Gamma^2\left(\frac{b}{2}\right) + \frac{\pi^2}{b^2}\left(\frac{b}{2}\right) \right] \right)$$

Because

$$\Gamma^2 + \frac{\pi^2}{b^2} = \frac{\omega^2}{c^2} - \frac{\pi^2}{b^2} + \frac{\pi^2}{b^2} = \frac{\omega^2}{c^2}$$

we get:

$$\frac{dPz}{dz} = -R_s \cdot \frac{A_{01}^{\ 2}}{\omega^2\mu^2} \cdot e^{-2\alpha z} \cdot \frac{b}{2} \cdot \frac{\omega^2}{c^2} \cdot \left(\frac{2a}{b}\frac{\pi^2}{b^2}\frac{c^2}{\omega^2} + 1\right)$$

or

$$\frac{dPz}{dz} = -R_s \cdot \frac{A_{01}^{\ 2}}{\eta^2} \cdot e^{-2\alpha z} \cdot \frac{b}{2} \cdot \left[\frac{2a}{b}\left(\frac{f_{c,01}}{f}\right)^2 + 1\right]$$

Remember:;

$$P(z) = \frac{1}{4}\frac{A_{01}^{\ 2}}{\eta}ab\sqrt{1-\left(\frac{f_{c,01}}{f}\right)^2} \cdot e^{-2\alpha z}$$

Now we can derive α as:

$$\alpha = -\frac{1}{2} \cdot \frac{\frac{dP(z)}{dz}}{P(z)} = -\frac{1}{2} \cdot \frac{-R_s \cdot \frac{A_{01}^{\ 2}}{\eta^2} \cdot e^{-2\alpha z} \cdot \frac{b}{2} \cdot \left[\frac{2a}{b}\left(\frac{f_{c,01}}{f}\right)^2 + 1\right]}{\frac{1}{4}\frac{A_{01}^{\ 2}}{\eta}ab\sqrt{1-\left(\frac{f_{c,01}}{f}\right)^2} \cdot e^{-2\alpha z}}$$

or finally:

$$\alpha_{TE,01} = R_s \cdot \frac{\left[\frac{2a}{b}\left(\frac{f_{c,01}}{f}\right)^2 + 1\right]}{\eta b \cdot \sqrt{1-\left(\frac{f_{c,01}}{f}\right)^2}} = \sqrt{\frac{\omega\mu}{2\sigma}} \cdot \frac{\left[\frac{2a}{b}\left(\frac{f_{c,01}}{f}\right)^2 + 1\right]}{\eta a \cdot \sqrt{1-\left(\frac{f_{c,01}}{f}\right)^2}}$$

This last expression is confirmed by the references:
- C. T. Johnk Engineering Electromagnetic Fields and Waves Wiley 1988, Chapter 8,p478.
- C. A. Balanis. Advanced Engineering Electromagnetics , Wiley 1989, Chapter 8,p379.

Remark that in both references the drivation of the attenuation is made for the mode TE10 instesad of TE01.
To go from one mode to the other, just simply permutate a and b in all expressions.